附錄 A5　三角函數的基本概念

A5-1

【類題練習 1】 $\sin B=\dfrac{2}{5}$, $\cos B=\dfrac{\sqrt{21}}{5}$, $\tan B=\dfrac{2\sqrt{21}}{21}$,

$\cot B=\dfrac{\sqrt{21}}{2}$, $\sec B=\dfrac{5\sqrt{21}}{21}$, $\csc B=\dfrac{5}{2}$

【類題練習 2】

$\angle A$	$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\csc A$
$30°$	$\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{3}}{3}$	$\sqrt{3}$	$\dfrac{2\sqrt{3}}{3}$	2
$45°$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$60°$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{2}$	$\sqrt{3}$	$\dfrac{\sqrt{3}}{3}$	2	$\dfrac{2\sqrt{3}}{3}$

【類題練習 3】 0

【家庭作業】

1. $\sin A=\dfrac{\sqrt{10}}{10}$, $\cos A=\dfrac{3\sqrt{10}}{10}$, $\cot A=3$,

   $\sec A=\dfrac{\sqrt{10}}{3}$, $\csc A=\sqrt{10}$

2. ① $(\sqrt{3}-1)/2$　② 0　③ 3/4　④ 0

3. 1

4. 75°

5. ① $\overline{BC}=12, \overline{CA}=16$

   ② $\cos A=\dfrac{4}{5}$, $\cos B=\dfrac{3}{5}$, $\tan A=\dfrac{3}{4}$, $\tan B=\dfrac{4}{3}$

6. $\csc A>\cot A>\sec A>\cos A>\tan A>\sin A$

7. 令 $\overline{BC}=a$, $\overline{CA}=b$, $\overline{AB}=c$, 其中 $a>b$.

   $\begin{cases} a^2+b^2=c^2 \\ a-b=\dfrac{c}{5} \end{cases}$, $\therefore c=5a-5b$ 代入

   $a^2+b^2=(5a-5b)^2=25a^2-50ab+25b^2$

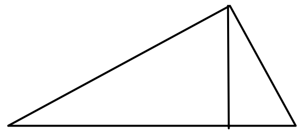
   $\therefore 24a^2-50ab+24b^2=0$

   $\Rightarrow\ (3a-4b)(4a-3b)=0$

$\Rightarrow \quad a=\frac{4}{3}b$ 或 $a=\frac{3}{4}b$ (不合)

$\therefore \tan A=\frac{a}{b}=\frac{4}{3}$

8. 如右圖，作 $\overline{AD}\perp\overline{BC}$.



$\because \overline{AB}=25$ 且 $\sin B=\frac{3}{5}$

$\therefore \overline{AB}:\overline{AD}=5:3=25:15\Rightarrow\overline{AD}=15$

又 $\because \overline{AD}=15$ 且 $\sin C=\frac{15}{17}$

$\therefore \overline{AC}:\overline{AD}=17:15\Rightarrow\overline{AC}=17$

依據畢氏定理，$\therefore \overline{AB}:\overline{AD}:\overline{BD}=5:3:4=25:15:20$

$\Rightarrow \overline{BD}=20$，且 $\overline{AC}:\overline{AD}:\overline{CD}=17:15:8\Rightarrow\overline{CD}=8$

$\therefore \overline{BC}=20+8=28$

因此 $\Delta ABC=\frac{1}{2}\times 28\times 15=210$.

9. 設 $\angle BOA=\angle EOD=\theta$.

$\because \tan\theta=\frac{\overline{DE}}{\overline{OD}}=\frac{\overline{DE}}{1}=\overline{DE}=\frac{12}{5}$,

$\therefore \frac{\overline{AB}}{\overline{OA}}=\tan\theta=\frac{12}{5}\Rightarrow\overline{AB}:\overline{OA}=12:5$.

依據畢氏定理，$\overline{OB}:\overline{AB}:\overline{OA}=13:12:5$.

$\therefore \frac{\overline{AB}}{\overline{OB}}=\sin\theta=\frac{12}{13}\Rightarrow\overline{AB}=\overline{OB}\times\frac{12}{13}=\frac{12}{13}$.

又 $\therefore \frac{\overline{OA}}{\overline{OB}}=\cos\theta=\frac{5}{13}\Rightarrow\overline{OA}=\overline{OB}\times\frac{5}{13}=\frac{5}{13}$

四邊形 OABC 的面積 $=\overline{OA}\times\overline{AB}=\frac{5}{13}\times\frac{12}{13}=\frac{60}{169}$

10. $\because \tan\angle BDC=\frac{3}{4}$, $\therefore \overline{DB}:\overline{CB}:\overline{CD}=4:3:5$

依據三角形外角定理，$\angle BDC=\angle A+\angle ACD=2\angle A$

$\therefore \angle ACD=\angle A\Rightarrow\Delta ACD$ 爲等腰三角形,

故 $\overline{AD}=\overline{CD}=5$.

$$\Rightarrow \tan A = \frac{\overline{BC}}{\overline{AB}} = \frac{3}{5+4} = \frac{1}{3}$$

A5-2

【類題練習 1】 $\cos A \cdot \sec A = 1$， $\tan A \cdot \cot A = 1$

【類題練習 2】 $\tan A = \cot B$, $\cot A = \tan B$; $\sec A = \csc B$, $\csc A = \sec B$

【類題練習 3】 $\frac{\cos A}{\sin A} = \cot A$, $\frac{\sec A}{\tan A} = \csc A$, $\frac{\tan A}{\sec A} = \sin A$,

$\frac{\csc A}{\cot A} = \sec A$, $\frac{\cot A}{\csc A} = \cos A$

【類題練習 4】 $\tan^2 A + 1 = \sec^2 A$, $\cot^2 A + 1 = \csc^2 A$

【家庭作業】

1. 1
2. 1
3. 1
4. $30°$
5. $$\frac{\sec A}{\sec A - \tan A} + \frac{\csc A}{1 - \csc A} = \frac{\frac{1}{\cos A}}{\frac{1}{\cos A} - \frac{\sin A}{\cos A}} + \frac{\frac{1}{\sin A}}{1 - \frac{1}{\sin A}}$$
$$= \frac{1}{1 - \sin A} + \frac{1}{\sin A - 1}$$
$$= 0$$
6. $\because \sin^2 A + \cos^2 A = 1$
$$\therefore (4\sin A + 3\cos A)^2 = 5^2 = 5^2(\sin^2 A + \cos^2 A)$$
$$\Rightarrow 16\sin^2 A + 24\sin A\cos A + 9\cos^2 A = 25\sin^2 A + 25\cos^2 A$$
$$\Rightarrow 9\sin^2 A - 24\sin A\cos A + 16\cos^2 A = 0$$
$$\Rightarrow (3\sin A - 4\cos A)^2 = 0$$
$$\Rightarrow 3\sin A - 4\cos A = 0$$
$$\Rightarrow \sin A = \frac{4}{3}\cos A$$
$$\Rightarrow \tan A = \frac{\sin A}{\cos A} = \frac{4}{3}$$
7. ① $$\frac{25}{12} = \tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$$
$$= \frac{\sin A \cdot \sin A}{\cos A \cdot \sin A} + \frac{\cos A \cdot \cos A}{\sin A \cdot \cos A}$$

$$=\frac{\sin^2 A+\cos^2 A}{\cos A\cdot\sin A}$$

$$=\frac{1}{\cos A\cdot\sin A}\Rightarrow \cos A\cdot\sin A=\frac{12}{25}$$

② $\because (\sin A+\cos A)^2$

$$=\sin^2 A+2\sin A\cos A+\cos^2 A$$
$$=\sin^2 A+\cos^2 A+2\sin A\cos A$$
$$=1+2\cdot\frac{12}{25}$$
$$=\frac{49}{25}$$

$\therefore\ \sin A+\cos A=\pm\sqrt{\frac{49}{25}}=\pm\frac{7}{5}$.(負數不合)

8. $\angle A$ 爲小於 $45°$ 的銳角,

所以 $90°-\angle A$ 也是小於 $45°$ 的銳角.

$\because$ sec 與 csc 具有餘角關係,

亦即當 $\sec A=\csc B$ 時, $\angle A+\angle B=90°$.

故 $\because \sec A=\csc(2A)$

$\therefore \angle A+2\angle A=90°\Rightarrow \angle A=30°$

9. $\frac{\sin A}{1+\cos A}+\frac{\sin A}{1-\cos A}$

$$=\frac{\sin A(1-\cos A)}{(1+\cos A)(1-\cos A)}+\frac{\sin A(1+\cos A)}{(1-\cos A)(1+\cos A)}$$
$$=\frac{\sin A-\sin A\cos A+\sin A+\sin A\cos A}{(1+\cos A)(1-\cos A)}$$
$$=\frac{2\sin A}{1-\cos^2 A}$$
$$=\frac{2\sin A}{\sin^2 A}$$
$$=\frac{2}{\sin A}$$
$$=2\csc A$$

A5-3

【類題練習 1】 $24\sqrt{2}/7$

p184【想想看】 (1) 雖然 90°不是銳角,但是 sin90°確實爲 1，也就是說，面積公式在直角三角形也是成立的.

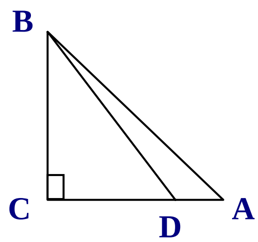
(2) 事實上，$\sin^2 A+\cos^2 A=1$對任意$\angle A$都是成立的，所以 $\cos 90°=\sqrt{1-\sin^2 90°}=0$.

【類題練習 2】 $3:6:4$

【家庭作業】 1. 如右圖，設$\overline{CD}=x$

$\because \angle BAC=45°$，$\therefore \overline{BC}=\overline{AC}=\overline{AD}+\overline{CD}=10+x$

$\because \angle BDC=60°$，$\therefore \overline{BC}:\overline{CD}=\sqrt{3}:1$



$\Rightarrow (10+x):x=\sqrt{3}:1$

$\Rightarrow \sqrt{3}x=10+x$

$\Rightarrow (\sqrt{3}-1)x=10$

$\Rightarrow x=\dfrac{10}{\sqrt{3}-1}=5(\sqrt{3}+1)$

即$\overline{DC}=5\sqrt{3}+5$

$\overline{\text{BC}}=10+5\sqrt{3}+5=15+5\sqrt{3}$

2. 先求$\Delta ABC=\dfrac{1}{2}\overline{AB}\times\overline{AC}\times\sin A=\dfrac{1}{2}\times 8\times 12\times\sin 60°$

$$=\frac{1}{2}\times 8\times 12\times\frac{\sqrt{3}}{2}=24\sqrt{3}$$

再設 $\overline{AD}=x$,

$\because \quad \Delta ADB+\Delta ADC=\Delta ABC$

$\therefore \quad \dfrac{1}{2}\overline{AB}\times\overline{AD}\times\sin\angle BAD+\dfrac{1}{2}\overline{AC}\times\overline{AD}\times\sin\angle CAD=24\sqrt{3}$

$\dfrac{1}{2}\cdot 8\cdot x\cdot\dfrac{1}{2}+\dfrac{1}{2}\cdot 12\cdot x\cdot\dfrac{1}{2}=24\sqrt{3}$

$\dfrac{1}{4}\cdot 20\cdot x=24\sqrt{3}$

$$x = \frac{24}{5}\sqrt{3} \Rightarrow \overline{AD} = \frac{24}{5}\sqrt{3}$$

3. 因為 O 為 ΔABC 的重心，所以 $\Delta OAB = \Delta OBC = \Delta OCA$.

設 $\overline{OD} : \overline{DA} = 1:2 = x:2x$

$\overline{OE} : \overline{EB} = 2:3 = 2y:3y$

$\overline{OF} : \overline{FC} = 3:4 = 3z:4z$

$$\Delta ODE : \Delta OAB = \overline{OD} \times \overline{OE} : \overline{OA} \times \overline{OB}$$
$$= x \cdot 2y : (x+2x) : (2y+3y) = 2:15$$

$$\Rightarrow \Delta ODE = \frac{2}{15}\Delta OAB$$

同理，$\Delta OEF = \dfrac{6}{35}\Delta OBC$，$\Delta OFD = \dfrac{1}{7}\Delta OCA$，

所以 ΔODE ： ΔOEF ： ΔOFD

$$= \frac{2}{15}\Delta OAB ： \frac{6}{35}\Delta OBC ： \frac{1}{7}\Delta OCA$$

$$= \frac{2}{15} : \frac{6}{35} : \frac{1}{7}$$

$$= 14:18:15$$